JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 32, 264–273 (1970)

A Population Process with Markovian Progenies

P. J. BROCKWELL* AND J. GANI†

Argonne National Laboratory
and
The Manchester-Sheffield School of Probability and Statistics, Sheffield, England

Submitted by S. Karlin

1. INTRODUCTION

Some recent results of Ali Khan and Gani [1, 3] on the first emptiness of an infinite dam fed by inputs forming a simple finite state Markov chain, have suggested the following more general problem for a discrete time population process.[1]

Consider an initial population of size $Z_0 = u$, at time $t = 0$, to which are added in each of the consecutive time intervals $(t, t + 1)$, the respective progenies $\{X_t\}$, $(t = 0, 1,...)$; the size of each such progeny depends on the size of the progeny in the previous time interval, so that

$$\Pr\{X_t = j \mid X_{t-1} = i\} = p_{ij} \geqslant 0 \qquad (i, j = 0, 1,...) \tag{1.1}$$

are the transition probabilities of a Markov chain with a denumerable infinity of states, and it is assumed for convenience that each $n \times n$ "northwest" truncation of the transition matrix is irreducible $(n = 1, 2, 3,...)$. At the end of each time interval, one individual in the total population dies or is removed, so that the total population at time t, after this removal is

$$\begin{aligned} Z_t &= Z_{t-1} + X_{t-1} - 1 \\ &= Z_0 + \sum_{n=0}^{t-1} (X_n - 1) \geqslant 0. \end{aligned} \tag{1.2}$$

* Research supported by the Office of Naval Research contract N0014-67-A-0112-0015 while the author was in the Department of Mathematics, Stanford University, Stanford, California.

† Work supported by the U.S. Office of Naval Research contract N00014-67-A-0112-0024 (NR-042-256) while the author was in the Department of Statistics, Stanford University, Stanford, California.

[1] The possibility of this generalization was raised by Dr. Peter Jagers at the Third Nordic Conference in Mathematical Statistics at Umea, Sweden, in June 1969.

The process comes to a stop at time T where $Z_T = 0$. It is already well known that the process $\{Z_t, X_{t-1}\}$ forms a bivariate Markov chain; we shall be concerned in this paper with the probabilities of extinction of the process $\{Z_t\}$ at times $T = u + j$ $(j = 0, 1,...)$.

In the case where the Markov chain has only a finite number of states $(i, j = 0, 1,..., r)$ and all $p_{ij} > 0$, Ali Khan and Gani [1] have shown that if $X_{-1} = 0$, then the probability generating function (p.g.f.)

$$G(\theta \mid u, 0) = \sum_{T=u}^{\infty} g(T \mid u, 0)\, \theta^T, \qquad (0 \leqslant \theta \leqslant 1) \tag{1.3}$$

of the extinction probabilities $g(T \mid u, 0)$ of the process at time T, given $Z_0 = u$, $X_{-1} = 0$, satisfies the equation

$$G(\theta \mid u, 0) = G_0{}^u(\theta) = \left\{ \sum_{T=1}^{\infty} g(T \mid 1, 0)\, \theta^T \right\}^u. \tag{1.4}$$

Here, the p.g.f. $G_0(\theta)$ is the solution of the functional equation

$$G_0(\theta) = \theta\lambda(G_0(\theta)) \tag{1.5}$$

where the function $\lambda(\theta)$ denotes the maximum eigenvalue of the positive irreducible matrix

$$P(\theta) = \{p_{ij}\theta^j\}_{i,j=0}^r, \qquad (0 < \theta \leqslant 1). \tag{1.6}$$

It has been shown in Gani [3] that $\lambda(\theta)$, though positive, simple, and strictly monotonic increasing in θ for $\theta > 0$ is not generally a p.g.f. The coefficients λ_i in its power expansion $\lambda(\theta) = \sum_{i=0}^{\infty} \lambda_i \theta^i$ are such that some λ_i may in fact be negative for $i \geqslant 2$. However, using Lagrange's theorem for the reversion of series, Eqs. (1.4)–(1.5) yield for the extinction probabilities $g(T \mid u, 0)$, the explicit form

$$g(T \mid u, 0) = \frac{u}{T} \lambda_{T-u}^{(T)}, \tag{1.7}$$

where

$$\lambda_{T-u}^{(T)} = \frac{\left\{ \left(\frac{d}{d\theta}\right)^{T-u} \lambda^T(\theta) \right\}_{\theta=0}}{(T-u)!}$$

is the coefficient of θ^{T-u} in $\lambda^T(\theta)$.

We proceed, in the following sections, to generalize these results to the population process $\{Z_t\}$ defined by (1.2), for which the progenies $\{X_t\}$ follow a simple Markov chain with a denumerable infinity of states whose transition probabilities (1.1) may be positive or zero. As already indicated we require only that the northwest truncations of the transition matrix be irreducible.

2. The p.g.f.'s $G_i(\theta)$ of the Extinction Probabilities when $Z_0 = 1$, $X_{-1} = i$

As in the earlier case of the finite state Markov chain, we shall denote by $g(T \mid u, i)$ the extinction probability

$$g(T \mid u, i) = \Pr\{Z_T = 0, \min_{0 \leqslant t < T} Z_t > 0 \mid Z_0 = u, X_{-1} = i\} \tag{2.1}$$

where $g(T \mid u, i)$ is meaningful, even if $u < i - 1$, where i may take any value $0 \leqslant i < \infty$.

Consider the probability $g(T \mid u, i)$; if $Z_T = 0$ so that the population becomes zero for the first time at $T \geqslant u$, then clearly at some prior time $j = u - 1, \ldots, T - 1$, the population $Z_j = 1$ for the first time. This cannot occur unless the progeny X_{j-1} in $(j - 1, j)$ is zero, so that

$$g(T \mid u, i) = \sum_{j=u-1}^{T-1} g(j \mid u - 1, i) g(T - j \mid 1, 0). \tag{2.2}$$

Let us now define the p.g.f.

$$G(\theta \mid u, i) = \sum_{T=u}^{\infty} g(T \mid u, i)\, \theta^T, \qquad (0 \leqslant \theta \leqslant 1).$$

Taking p.g.f.'s with respect to time in (2.2) we obtain

$$G(\theta \mid u, i) = G(\theta \mid u - 1, i)\, G(\theta \mid 1, 0) = G(\theta \mid u - 1, i)\, G_0(\theta) \tag{2.3}$$

where $G(\theta \mid 1, 0)$ is denoted by $G_0(\theta)$ for simplicity. Repeating this procedure $u - 1$ times, we find that

$$G(\theta \mid u, i) = G(\theta \mid 1, i)\, G_0^{u-1}(\theta) = G_i(\theta)\, G_0^{u-1}(\theta) \tag{2.4}$$

or, if $X_{-1} = 0$,

$$G(\theta \mid u, 0) = G_0{}^u(\theta) \tag{2.5}$$

precisely as in (1.4).

Let us now consider in detail the derivation of the equation satisfied by $G_i(\theta) = G(\theta \mid 1, i)$. Suppose that in the time interval $(0, 1)$ the progeny

$X_0 = j \geqslant 0$ is added to $Z_0 = 1$; then from (1.2) $Z_1 = Z_0 + j - 1 = j$, so that extinction will follow thereafter with p.g.f.

$$G(\theta \mid j, j) = G(\theta \mid 1, j)\, G_0^{j-1}(\theta).$$

Thus, it is clear that

$$G(\theta \mid 1, i) = \theta \sum_{j=0}^{\infty} p_{ij} G(\theta \mid 1, j)\, G_0^{j-1}(\theta) \tag{2.6}$$

or more simply

$$G_i(\theta) = \frac{\theta}{G_0(\theta)} \sum_{j=0}^{\infty} p_{ij} G_j(\theta)\, G_0{}^j(\theta). \tag{2.7}$$

By its definition, $G_i(\theta)$ will be zero for $\theta = 0$, strictly positive in $0 < \theta \leqslant 1$ by virtue of the irreducibility of the chain, and monotonic increasing in $0 < \theta \leqslant 1$ with $G_i(1) \leqslant 1$. Let $G(\theta)$ be the infinite column vector defined by

$$G(\theta) = \{G_0(\theta), G_1(\theta),\ldots, G_i(\theta),\ldots\}',$$

then from (2.7)

$$\left[I - \frac{\theta}{G_0(\theta)} P \operatorname{diag}\{G_0{}^j(\theta)\}\right] G(\theta) = 0 \tag{2.8}$$

where I is the infinite unit matrix, and P the infinite matrix of transition probabilities $\{p_{ij}\}_{i,j=0}^{\infty}$ defined by (1.1). We may simplify (2.8) by writing it in the form

$$G = \frac{\theta}{G_0(\theta)} MG \tag{2.9}$$

where $G = G(\theta)$ and M denotes the well-defined infinite matrix with non-negative elements

$$M_{ij} = p_{ij} G_0{}^j(\theta) = p_{ij} G_0{}^j. \tag{2.10}$$

In order to solve for the basic p.g.f. $G_0(\theta)$, it is necessary to consider operations on this matrix; this we proceed to do.

3. The Equation for $G_0(\theta)$

Consider the population whose progeny sizes are Markovian with transition matrix $P_{(n)}$, the northwest $(n+1) \times (n+1)$ truncation of P. Unlike P, the truncated matrix $P_{(n)}$ is not in general stochastic. The deficiency $[1 - \sum_{j=0}^{n} p_{ij}]$ of a typical row sum of $P_{(n)}$ from unity we may interpret as

$\Pr\{X_{t+1} = \infty \mid X_t = i\}$. This new process will be referred to as the *truncated process.*

As in Section 2 we define the p.g. functions

$$G_{(n)}(\theta \mid u, i) = \sum_{T=u}^{\infty} g_{(n)}(T \mid u, i)\, \theta^T, \qquad 0 \leqslant \theta \leqslant 1, \qquad i = 0, 1, \ldots, n$$

where $g_{(n)}(T \mid u, i)$ is the probability of extinction at time T for the truncated process given that the population size at time zero is u and $X_{-1} = i$. Notice that if an infinite progeny size occurs in the truncated process, then extinction is not possible at any finite time.

Arguing precisely as in Section 2 we find that the vector

$$G_{(n)}(\theta) = \{G_{0(n)}(\theta), G_{1(n)}(\theta), \ldots, G_{n(n)}(\theta)\}'$$

of generating functions $G_{i(n)}(\theta) = G_{(n)}(\theta \mid 1, i)$ satisfies the equation

$$G_{(n)}(\theta) = \frac{\theta}{G_{0(n)}(\theta)} M_{(n)} G_{(n)}(\theta) \tag{3.1}$$

where $M_{(n)}$ is the $(n + 1) \times (n + 1)$ matrix whose elements are given by

$$M_{(n)ij} = p_{ij} G_{0(n)}^{j}(\theta), \qquad i, j = 0, 1, \ldots, n. \tag{3.2}$$

Since it has been assumed that each of the truncations $P_{(n)}$, $n = 0, 1, \ldots$ is irreducible it follows that for $0 < \theta \leqslant 1$ the vectors $G_{(n)}(\theta)$ are strictly positive eigenvectors of the nonnegative matrices $M_{(n)}$. A result from the Perron-Frobenius theory of nonnegative matrices then implies that the eigenvalue of $M_{(n)}$ corresponding to $G_{(n)}(\theta)$ is precisely the spectral radius of $M_{(n)}$. Denoting this spectral radius by $\lambda(M_{(n)})$ we have from Eq. (3.1) that

$$\lambda(M_{(n)}) = \theta^{-1} G_{0(n)}(\theta), \qquad 0 < \theta \leqslant 1. \tag{3.3}$$

Equations (3.2) and (3.3) uniquely determine $G_{0(n)}(\theta)$ in the range $0 < \theta \leqslant 1$.

Now we know that the extinction probabilities $g_{(n)}(T \mid 1, 0)$ for the truncated process can be written

$$g_{(n)}(T \mid 1, 0) = \sum_{D_n} p_{0 j_0} p_{j_0 j_1} \cdots p_{j_{T-2} j_{T-1}} \tag{3.4}$$

where, as it is usually stated in the literature, the summation is over the set

$$D_n = \left\{ (j_0, j_1, \ldots, j_{T-1}) : 0 \leqslant j_0, j_1, \ldots, j_{T-1} \leqslant n; \right.$$
$$\left. \sum_{i=k}^{T-1} j_i \leqslant T - 1 - k, k = 1, \ldots, T - 1; \sum_{i=0}^{T-1} j_i = T - 1 \right\}. \tag{3.5}$$

Hence $g_{(n)}(T \mid 1, 0)$ is nondecreasing in n for each T and moreover

$$g_{(n)}(T \mid 1, 0) = g(T \mid 1, 0) \qquad \text{for} \qquad n \geqslant T - 1 \tag{3.6}$$

where $g(T \mid 1, 0)$ is the extinction probability for the nontruncated process. It follows that the generating function

$$G_0(\theta) = \sum_{T=1}^{\infty} g(T \mid 1, 0)\, \theta^T$$

is given by

$$G_0(\theta) = \lim_{n\to\infty} G_{0(n)}(\theta), \qquad |\theta| \leqslant 1. \tag{3.7}$$

Thus $G_0(\theta)$ is uniquely determined in the range $0 < \theta \leqslant 1$ by Eqs. (3.2), (3.3), and (3.7).

Equations (3.2) and (3.3) define $G_{0(n)}$ as a function of θ with domain $0 < \theta \leqslant 1$. Equivalently they may be regarded as defining a function θ_n of G with domain $0 < G \leqslant G_{0(n)}(1)$. Using the latter interpretation we can rewrite (3.3) as

$$G = \theta_n(G)\, \lambda(M_{(n)}(G)), \qquad 0 < G \leqslant G_{0(n)}(1). \tag{3.8}$$

Letting n tend to ∞ in Eq. (3.8) we obtain, using a result of Seneta [4], that

$$G = \lim_{n\to\infty} \theta_n(G)\, \lambda(M(G)), \qquad 0 < G \leqslant G_0(1) \tag{3.9}$$

where $\lambda(M(G))$ is the convergence norm (as defined by Vere-Jones [5, 6]) of the infinite matrix with elements

$$M_{ij}(G) = p_{ij} G^j, \qquad i, j = 0, 1, 2, \ldots .$$

But $\lim_{n\to\infty} \theta_n(G)$ is the inverse function of $G_0(\theta) = \lim_{n\to\infty} G_{0(n)}(\theta)$. Hence we can rewrite Eq. (3.9) as

$$G_0(\theta) = \theta\lambda(M(G_0(\theta))), \qquad 0 < \theta \leqslant 1, \tag{3.10}$$

where $\lambda(M(G_0(\theta)))$ is the convergence norm of the matrix $M(G_0(\theta))$. This is the analogue, for the infinite matrix case, of the result of Ali Khan and Gani [1].

4. A Power-Series Expansion of $\lambda(M)$ and an Algorithm for its Coefficients

The purpose of this section is to show that the convergence norm $\lambda(M(w))$ of any matrix $M(w)$ having the form (2.11), i.e., with elements

$$M_{ij} = p_{ij} w^j, \qquad i, j = 0, 1, 2, \ldots, \tag{4.1}$$

can be expanded as a power series in a neighborhood of $w = 0$. Moreover, an algorithm will be developed for determining the consecutive coefficients in this expansion.

Since $\theta^{-1}G_0(\theta)$ is a p.g.f. it is clear from Eq. (3.10) that $\lambda(M(G_0(\theta)))$ can be extended by analytic continuation to an analytic function of θ in the complex domain $\{\theta : |\theta| < 1\}$.

Consider the equation

$$w = G_0(\theta) = \theta \sum_{T=1}^{\infty} g(T \mid 1, 0)\, \theta^{T-1}, \qquad |\theta| < 1. \tag{4.2}$$

Since $\lambda(M(G_0(\theta)))$ is an analytic function of θ for $|\theta| < 1$ it will be sufficient, for analyticity of $\lambda(M(w))$ in a neighborhood of $w = 0$, to show that there is a neighborhood $\{w : |w| < \delta\}$ which is mapped analytically by the relation (4.2) into the interior of the unit disc $\{\theta : |\theta| < 1\}$. To show this we write (4.2) in the form

$$\theta = w\varphi(\theta), \tag{4.3}$$

where

$$\varphi(\theta) = \left[\sum_{T=1}^{\infty} g(T(1, 0)\, \theta^{T-1}\right]^{-1}. \tag{4.4}$$

By Lagrange's theorem (Copson [2], pp. 123-125), since $\theta/\varphi(\theta)$ is analytic in a neighborhood of $\theta = 0$ and $d/d\theta(\theta/\varphi(\theta)) = g_1 \neq 0$ when $\theta = 0$, it follows that Eq. (4.3) has a unique solution $\theta(w)$ in a neighborhood $N_\delta = \{w : |w| < \delta\}$ of $w = 0$, and moreover, that the solution is analytic in this neighborhood. Since $\theta(0) = 0$ it is clear that by choosing δ sufficiently small we can ensure that N_δ is mapped analytically into $\{\theta : |\theta| < 1\}$ as required. Thus we have shown that $\lambda(M(w))$ is analytic in the neighborhood N_δ of $w = 0$.

We now proceed to determine the coefficients λ_i in the expansion

$$\lambda(M(w)) = \sum_{i=0}^{\infty} \lambda_i w^i, \qquad |w| < \delta. \tag{4.5}$$

Consider the truncated matrix $M_{(n)}$ given by

$$M_{(n)} = \begin{bmatrix} p_{00} & p_{01}w & p_{02}w^2 & \cdots & p_{0n}w^n \\ \vdots & & & & \vdots \\ p_{n0} & p_{n1}w & p_{n2}w^2 & \cdots & p_{nn}w^n \end{bmatrix}, \qquad w > 0, \tag{4.6}$$

and denote its spectral radius (for w sufficiently small) by

$$\lambda_{(n)}(w) = \sum_{i=0}^{\infty} \lambda_{(n)i} w^i. \tag{4.7}$$

(The fact that $\lambda_{(n)}(w)$ can be extended to a function analytic in a neighborhood of $w = 0$ is proved exactly as for the infinite matrix case above) We now show that

$$\lambda_i = \lambda_{(n)i}, \qquad i = 0, 1, \dots, n; \quad n = 0, 1, 2, \dots . \tag{4.8}$$

This follows from the fact that the extinction probabilities $\{g_{(n)}(T \mid 1, 0)\}_{T=1}^{n+1}$ of the truncated process are in one-to-one correspondence with the coefficients $\lambda_{(n)i}$, $i = 0, 1, \dots, n$. This correspondence is given by

$$Tg_{(n)}(T \mid 1, 0) = \lambda_{(n)T-1}^{(T)}, \tag{4.9}$$

where $\lambda_{(n)T-1}^{(T)}$ is the coefficient of w^{T-1} in $[\lambda_{(n)}(w)]^T$. (Rewriting equation (4.9) as

$$\frac{1}{2\pi i T} \oint \frac{\lambda_{(n)}^T(z)}{z^T}\, dz = g_{(n)}(T \mid 1, 0),$$

where the contour of integration is a sufficiently small circle enclosing $z = 0$, we can exhibit $G_{0(n)}(\theta)$ in the succinct form

$$G_{0(n)}(\theta) = \frac{1}{2\pi i} \oint \ln(1 - z^{-1}\lambda_{(n)}(z)\, \theta)\, dz).$$

A relation exactly analogous to (4.9) holds also for the extinction probabilities $g(T \mid 1, 0)$ of the nontruncated process, viz.

$$Tg(T \mid 1, 0) = \lambda_{T-1}^{(T)} . \tag{4.10}$$

However have already seen from (3.6) that $g_{(n)}(T \mid 1, 0) = g(T \mid 1, 0)$ for $n \geqslant T - 1$ so that

$$\lambda_{T-1}^{(T)} = \lambda_{(n)T-1}^{(T)}, \qquad 0 \leqslant T - 1 \leqslant n, \tag{4.11}$$

and the required result (4.8) follows.

Using Eq. (4.8) and the fact that $\lambda_{(n)}(w)$ satisfies the equation

$$| M_{(n)} - I_{(n)}\lambda_{(n)}(w)| = 0, \qquad n = 0, 1, 2, \dots, \tag{4.12}$$

[where $I_{(n)}$ is the $(n + 1) \times (n + 1)$ indentity matrix], we obtain the following recursion relations for λ_n:

$$\lambda_0 = p_{00}, \tag{4.13}$$

$$\begin{aligned}(-p_{00})^{n-1}\lambda_n = &-p_{0n}(-p_{00})^{n-2}p_{n0} \\ &+ \text{coefficient of } w^n \text{ in } \left| M_{(n-1)} - I_{(n-1)} \sum_{j=0}^{n-1} \lambda_j w^j \right|, \\ n = &\ 1, 2, \dots .\end{aligned} \tag{4.14}$$

These relations follow from the requirement that the coefficient of w^n in the left hand side of (4.12) be zero.

5. An Example and Some Concluding Remarks

For the particular infinite transition matrix

$$P = \begin{bmatrix} p+q & r & & & & \\ p & q & r & & & \\ & p & q & r & & \\ & & p & q & r & \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot \ \cdot \end{bmatrix}, \qquad (p+q+r=1; p, r > 0; q \geqslant 0), \tag{5.1}$$

the recurrence relations (4.13) and (4.14) lead to the following coefficients in the expansion of $\lambda(M(w))$:

$$\begin{aligned} \lambda_0 &= \alpha, \\ \lambda_1 &= \alpha^{-1}\beta, \\ \lambda_2 &= \alpha^{-2}\beta q - \alpha^{-3}\beta^2, \\ \lambda_3 &= \alpha^{-3}\beta q^2 - 3\alpha^{-4}\beta^2 q + 2\alpha^{-5}\beta^3, \\ \lambda_4 &= \alpha^{-3}\beta^2 + \alpha^{-4}\beta q^3 - 6\alpha^{-5}\beta^2 q^2 + 10\alpha^{-6}\beta^3 q - 5\alpha^{-7}\beta^4, \\ \lambda_5 &= 2\alpha^{-4}\beta^2 q + \alpha^{-5}(\beta q^4 - 4\beta^3) \\ &\quad - 10\alpha^{-6}\beta^2 q^3 + 30\alpha^{-7}\beta^3 q^2 - 35\alpha^{-8}\beta^4 q + 14\alpha^{-9}\beta^5, \end{aligned} \tag{5.2}$$

where $\alpha = p + q$ and $\beta = pr$. The extinction probabilities are obtained from the coefficients λ_i by means of equation (4.10). Thus we find

$$\begin{aligned} &g(1 \mid 1, 0) = p + q, && g(4 \mid 1, 0) = pq^2r, \\ &g(2 \mid 1, 0) = pr, && g(5 \mid 1, 0) = pq^3r + p^3r^2 + p^2r^2q, \\ &g(3 \mid 1, 0) = pqr, && g(6 \mid 1, 0) = pq^4r + 2p^2q^2r^2 + 2p^3qr^2 + p^3r^3. \end{aligned} \tag{5.3}$$

Repeated applications of Eqs. (4.13), (4.14), and (4.10) enable one to determine recursively the probabilities $g(T \mid 1, 0)$ for larger values of T.

In the case of the truncated process it is possible to obtain the extinction probabilities $g_{(n)}(T \mid 1, j)$ by a matrix inversion. Thus

$$G_{(n)}(\theta) = \theta[I - \theta P_{1(n)} \operatorname{diag}\{G^j_{0(n)}(\theta)\}]^{-1} p_{0(n)} \tag{5.4}$$

where $G_{(n)}(\theta)$ is the vector previously defined in Section 3, $P_{1(n)} = \{p_{ij}\}^n_{i,j=1}$ and $p'_{0(n)} = \{p_{10}, p_{20}, \ldots, p_{n0}\}$. Since for $T \leqslant n - 1$ we have

$$g_{(n)}(T \mid 1, j) = g(T \mid 1, j)$$

it follows that the latter can be obtained for any T from Eq. (5.4) provided n is chosen such that $n \geqslant T + 1$.

We note once again as in Gani [3] that the coefficients $\lambda_{(n)i}$ and λ_i may be negative for $i \geqslant 2$; this is certainly the case for λ_2 in the example above if $r(1 - r) > q$. It would be of interest to investigate further the reason why some of the λ_i are negative while $\lambda_{T-1}^{(T)}$ is always positive.

It is proposed to study the continuous time analogue of this population model with a view to ascertaining whether parallel results can be derived.

References

1. M. S. Ali Khan and J. Gani, Infinite dams with inputs forming a Markov chain, *J. Appl. Probability* **5** (1968), 72–83.
2. E. T. Copson, "An Introduction to the Theory of Functions of a Complex Variable," Oxford Univ. Press, London/New York, 1935.
3. J. Gani, A note on the first emptiness of dams with Markovian inputs, *J. Math. Anal. Appl.* **26** (1969), 270–274.
4. E. Seneta, Finite approximations to infinite non-negative matrices, *Proc. Cambridge Philos. Soc.* **63** (1967), 983–992.
5. D. Vere–Jones, Ergodic properties of non-negative matrices, I, *Pacific J. Math.* **22** (1967), 361–386.
6. D. Vere–Jones, Ergodic properties of non-negative matrices, II, *Pacific J. Math.* **26** (1968), 601–620.